

Cubic Spline Example

Determine the parameters a, b, c, d, e, f, g and h so that $S(x)$ is a natural cubic spline where

$$S(x) = \begin{cases} ax^3 + bx^2 + cx + d & x \in [-1, 0] \\ ex^3 + fx^2 + gx + h & x \in [0, 1] \end{cases}$$

with interpolation conditions $S(-1) = 1, S(0) = 2$ and $S(1) = -1$.

On $[-1, 0], S(x) = ax^3 + bx^2 + cx + d$

On $[0, 1], S(x) = ex^3 + fx^2 + gx + h$

Continuity conditions: $S(x), S'(x)$ and $S''(x)$ continuous at $x = 0$.

End conditions: $S''(x) < 0$ at $x = -1, x = +1$.

$$S'(x) = \begin{cases} 3ax^2 + 2bx + c \\ 3ex^2 + 2fx + g \end{cases} \quad \text{and} \quad S''(x) = \begin{cases} 6ax + 2b \\ 6ex + 2f \end{cases}$$

At $x = 0, \quad S(x) = 2 \Rightarrow d = h = 2 \quad (1)$

$S'(x)$ continuous $\Rightarrow c = g \quad (2)$

$S''(x)$ continuous $\Rightarrow b = f \quad (3)$

At $x = -1, \quad S''(x) = 0 \Rightarrow -6a + 2b = 0 \Rightarrow b = 3a \quad (4)$

At $x = +1, \quad S''(x) = 0 \Rightarrow 6e + 2f = 0 \Rightarrow f = -3e \quad (5)$

From (3), (4) and (5): $a = -e$

Also, at $x = -1, S(x) = 1 \quad \Rightarrow -a + b - c + d = 1 \Rightarrow -a + b - c = -1 \quad (6)$

And at $x = +1, S(x) = -1 \quad \Rightarrow e + f + g + 2 = -1 \Rightarrow -a + b + c = -3 \quad (7)$

From (6) and (7): $c = -1$; from (7): $-a + b = -2$; and from (4): $-a + 3a = -2 \Rightarrow a = -1$

Hence $a = -1, b = 3a = -3, c = -1, d = 2$

$e = -a = +1, f = b = -3, g = c = -1, h = d = 2$.